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# FORCED PERIODIC OSCILLATIONS OF THE SIMPLE PENDULUM

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Periodic solutions of second order differential equation that define oscillations of a simple pendulum subjected to an external sinusoidal force are considered. The class of symmetric periodic solutions that satisfy boundary conditions is determined in the case of small amplitude of the acting force. These solutions are extended to the region of large amplitudes of the acting force, using numerical computations. Branching of obtained solutions is investigated.

1. Periodic oscillations at small amplitudes of the acting force. Let us consider the differential equation

$$x^{\cdot\cdot} + \mu \sin x = e \sin t \tag{1.1}$$

where $x$ is the unknown function, $t$ is the independent variable, and $e$ and $\mu$ are parameters. This equation may be considered to be the equation of motion of a simple pendulum subjected to an external sinusoidal force. We seek a periodic solution of this equation that for $e = 0$ would coincide with the periodic solutions of the corresponding homogeneous equation.

Using the Poincaré method of the small parameter [1] it is possible to show that for reasonably small $|e|$ and $\mu \neq l^2$ $(l = 0, 1, \ldots)$ Eq. (1.1) has a unique $2\pi$-periodic solution $x_z(t, e)$ which analytically depends on $e$ and vanishes for $e = 0$. The following relationships are valid for that solution:

$$x_z(t + \pi, e) = -x_z(t, e), \quad x_z(-t, e) = -x_z(t, e) \tag{1.2}$$

$$x_z(t, -e) = -x_z(t, e)$$

The numerical derivation of solution $x_z$ reduces to solving for Eq. (1.1) of the boundary value problem

$$x(0) = x^{\cdot}(\pi/2) = 0 \tag{1.3}$$

Let $m$ and $n$ be relatively prime natural numbers and $\mu > 0$. We shall find the $2\pi m$- periodic solution of Eq. (1.1), which for $e = 0$ is transformed into $2\pi m/n$-periodic solution of the homogeneous equation. The sought solutions are called solutions of the $m / n$ form [2]. We use the methods proposed in [3, 4] for determining such solutions, and introduce function $x_0(t) = 2 \arcsin (k \operatorname{sn} \sqrt{\mu} t)$ in which the module of elliptic functions $k$ is the root of equation $2n\mathbf{K}(k) = \pi m \sqrt{\mu}$, and $\mathbf{K}(k)$ is a complete elliptic integral of the first kind. Function $x_0(t)$ is determinate for $\mu > n^2 / m^2$, satisfies Eq. (1.1), and for $e = 0$ is $2\pi m / n$-periodic.

Let us consider the auxilliary system

$$x^{\cdot\cdot} + \mu \sin x = e \sin t - p x_0^{\cdot}(t + t_0) \tag{1.4}$$

$$\int_0^{2\pi m} x x_0^{\cdot}(t + t_0)\, dt = 0$$

where $x$ is the unkown function, $p$ is the unknown constant, and $t_0$ is an arbitrary constant. Using the results of [3] it is possible to show that for reasonably small $|e|$ system (1.4) has the unique $2\pi m$-periodic with respect to $t$ solution

$$x = x_* (t, t_0, e), \quad p = p_* (t_0, e) \tag{1.5}$$

whose dependence on $e$ is analytic, and which satisfies conditions $x_* (t, t_0, 0) = x_0 (t + t_0)$ and $p_* (t_0, 0) = 0$. Let $t_0 (e)$ the root of the so-called bifurcation equation

$$p_* (t_0, e) = 0. \tag{1.6}$$

Then $x (t, e) = x_* [t, t_0 (e), e]$ is the periodic solution of the $m / n$ form of Eq. (1.1). The converse statement is also valid. The determination of periodic solutions of the $m/n$ form thus reduces to the determination of roots of Eqs. (1.6).

Let us point out some of the properties of solutions (1.5). Using the obvious equalities $x_0 (t + \pi m/n) = -x_0 (t)$ and $\sin (t + \pi) = -\sin t$ we can show that

$$x_* (t + \pi, t_0, e) = x_* (t, t_0 + \pi, -e), \tag{1.7}$$
$$x_* \left(t, t_0 + \frac{\pi m}{n}, e\right) = -x_* (t, t_0, -e)$$

$$p_* (t_0 + \pi, e) = p_* (t_0, -e), \quad p_* \left(t_0 + \frac{\pi m}{n}, e\right) = p_* (t_0, -e) \tag{1.8}$$

Since by virtue of (1.7) $x_* (t, t_0 + 2\pi m / n, e) = x_* (t, t_0, e)$, hence it is sufficient to determine the roots of Eq. (1.6) in the interval $0 \leqslant t_0 < 2\pi m / n$. Function $p_* (t_0, e)$ is periodic of period $2\pi / n$ with respect to $t_0$. Since $m$ and $n$ are relatively prime natural numbers, there exist integers $s_1$ and $s_2$ such that

$$s_1 m + s_2 n = 1 \tag{1.9}$$

Hence $2\pi / n = 2\pi m s_1 / n + 2\pi s_2$ and in consequence of (1.8)

$$p_* \left(t_0 + \frac{2\pi}{n}, e\right) = p_* (t_0, e) \tag{1.10}$$

Using the oddness of functions $x_0 (t)$ and $\sin t$ it is possible to establish the relationships

$$x_* (t, -t_0, e) = -x_* (-t, t_0, e), \quad p_* (-t_0, e) = -p_* (t_0, e) \tag{1.11}$$

On the strength of the last of these and of formula (1.10) Eq. (1.6) has the trivial roots

$$t_0^{(r)} = \pi r/n \quad (r = 0, 1, \ldots, 2m - 1) \tag{1.12}$$

to which correspond periodic solutions of the $m / n$ type

$$x_{n/m}^{(r)} (t, e) = x_* (t, t_0^{(r)}, e) \quad (r = 0, 1, \ldots, 2m - 1) \tag{1.13}$$

Using formulas (1.7), (1.8), and (1.11) we obtain the equalities

$$x_{n/m}^{(s)} (t, e) = x_{n/m}^{(r)} (t + \pi, -e) \tag{1.14}$$
$$-x_{n/m}^{(r)} (-t + a_r, e) = x_{n/m}^{(r)} (t + a_r, e)$$
$$s = r + n \ (\mathrm{mod}\ 2m), \ a_r = -\pi r s_2 \ (\mathrm{mod}\ \pi m)$$

where $s_2$ is an integer which with some other integer $s_1$ satisfies formula (1.9).

Further analysis of periodic solutions of the $m / n$ type is based on the parity properties of numbers $m$ and $n$. Owing to the relative primality of these numbers two cases are possible: 1) both $m$ and $n$ are odd, 2) one of these numbers is even, the other odd. Let us first consider case 1). Using formulas (1.7) and the oddness of $m$ and $n$, we obtain

$$x_{n/m}^{(r)}(t+\pi m, e) = -x_{n/m}^{(r)}(t, e)$$

$$x_{n/m}^{(s)}(t, e) = -x_{n/m}^{(r)}(t, -e) \quad (s = r + m \pmod{2m})$$

Thus solutions (1.13) are $\pi m$-antiperiodic. Because of (1.14) it is sufficient for the derivation of all solutions (1.13) to determine solutions $x_{n/m}^{(0)}$ and $x_{n/m}^{(m)}$ for $e \geqslant 0$ which are determined by the boundary conditions $x(0) = x^{\cdot}(\pi m / 2) = 0$.

Let now one of the numbers $m$ and $n$ be even. In that case the integers $s_1$ and $s_2$ in (1.9) can be selected odd. Let us assume that their selection conforms to this. Owing to the relationship $\pi / n = \pi m s_1 / n + \pi s_2$, formulas (1.8), and the oddness of numbers $s_1$ and $s_2$ we have $p_*(t_0 + \pi / n, e) = p_*(t_0, e)$. From which, taking into account (1.11) we find that in this case Eq. (1.6) has in addition to roots (1.12) the trivial roots

$$\bar{t}_0^{(r)} = \frac{\pi(2r+1)}{2n} \quad (r = 0, 1, \ldots, 2m-1)$$

To these roots correspond periodic solutions of the $m/n$ type

$$\bar{x}_{n/m}^{(r)}(t, e) = x_*(t, \bar{t}_0^{(r)}, e) \quad (r = 0, 1, \ldots, 2m-1) \tag{1.15}$$

for which the following relationships are valid:

$$\bar{x}_{n/m}^{(s)}(t, e) = \bar{x}_{n/m}^{(r)}(t + \pi, -e)$$

$$\bar{x}_{n/m}^{(r)}(-t + \bar{a}_r, e) = \bar{x}_{n/m}^{(r)}(t + \bar{a}_r, e)$$

$$s = r + n \pmod{2m}, \quad \bar{a}_r = -\pi(2r+1)s_2 / 2 \pmod{\pi m}$$

On the strength of (1.7), (1.9), and oddness of $s_2$ we have

$$x_{n/m}^{s}(t, e) = -x_{n/m}^{(r)}(t + \pi s_2, e) \tag{1.16}$$

$$\bar{x}_{n/m}^{(s)}(t, e) = -\bar{x}_{n/m}^{(r)}(t + \pi s_2, e), \quad s = r + 1 \pmod{2m}$$

Owing to relationships (1.14) and (1.6) it is sufficient for the determination of all solutions (1.13) to find solution $x_{n/m}^{(0)}$ for $e \geqslant 0$ The numerical determination of that solution reduces to solving the boundary value problem $x(0) = x(\pi m) = 0$ for Eq. (1.1). Exactly in the same way for obtaining all solutions (1.15) it is sufficient to find solution $\bar{x}_{n/m}^{(q)}$, where $0 \leqslant q < m$, $(2q+1)s_2 = -1 \pmod{2m}$ for $e \geqslant 0$. That solution is determined by the boundary conditions $x^{\cdot}(\pi/2) = x^{\cdot}(\pi/2 + \pi m) = 0$.

The form of Eq. (1.1) implies that in addition to periodic solutions $x_z$ (1.13) and (1.15) that equation has periodic solutions derived from the indicated [equations] using the transformation $x \to x + \pi$, and $\mu \to -\mu$. Solutions $x_z$ (1.13) and (1.15) were obtained above for $|e| \ll 1$. By solving the respective boundary value problems these solutions can be continued in the region of considerable $|e|$. In this way $2\pi$-periodic solutions that coincide for $e \ll 1$ with solutions $x_z$, $x_{1/1}^{(r)}$, $x_{2/1}^{(r)}$, $\bar{x}_{2/1}^{(r)}$, and $x_{3/1}^{(r)}$ $(r = 0, 1)$ in region $E = \{e, \mu : 0 \leqslant e \leqslant 10, |\mu| \leqslant 10\}$. A brief description of results of the described investigation is given below. The method of computation is that described in [5, 6].

2. $2\pi$-periodic solutions. Let $n$ be an odd integer. Solutions $x^{(0)}_{n/1}$ and $x^{(1)}_{n/1}$ determined for $|e| \ll 1$ and $\mu > n^2$ are odd $\pi$-antiperiodic functions of $t$. The numerical derivation of these solutions reduces to solving the boundary value problem (1.1), (1.3). It can be shown that any solution of such problem is odd and $\pi$-antiperiodic.

If $n$ is even then for $|e| \ll 1$ and $\mu > n^2$ the solutions $x^{(0)}_{n/1}$, $x^{(1)}_{n/1}$, $\bar{x}^{(0)}_{n/1}$, and $\bar{x}^{(1)}_{n/1}$ exist, and for them the following equalities are valid:

$$x^{(r)}_{n/1}(-t, e) = -x^{(r)}_{n/1}(t, e), \quad \bar{x}^{(r)}_{n/1}\left(-t + \frac{\pi}{2}, e\right) = \bar{x}^{(r)}_{n/1}\left(t + \frac{\pi}{2}, e\right) \quad (r = 0, 1) \tag{2.1}$$

$$x^{(0)}_{n/1}(t + \pi, e) = -x^{(1)}_{n/1}(t, e), \quad \bar{x}^{(0)}_{n/1}(t + \pi, e) = -\bar{x}^{(1)}_{n/1}(t, e)$$

The solutions $x^{(0)}_{n/1}$ and $x^{(1)}_{n/1}$ are determined by the boundary conditions

$$x(0) = x(\pi) = 0 \tag{2.2}$$

and solutions $\bar{x}^{(0)}_{n/1}$ and $\bar{x}^{(1)}_{n/1}$ by the boundary conditions

$$\dot{x}(\pi/2) = \dot{x}(3\pi/2) = 0 \tag{2.3}$$

These conditions are also satisfied by solutions $x_z$ and $x^{(0)}_{n/1}$ and $x^{(1)}_{n/1}$ for odd $n$. It can be shown that any solution of the boundary value problem (1.1), (2.2) is odd and $2\pi$-periodic, and that any solution of the boundary value problem (1.1), (2.3) is $2\pi$-periodic and satisfies the relationship $x(-t + \pi/2) = x(t + \pi/2)$.

In region $E$ with $e \ll 1$ exists solution $x_z$ and eight solutions $1/n$: $x^{(r)}_{1/1}$, $x^{(r)}_{2/1}$, $\bar{x}^{(r)}_{2/1}$, and $x^{(r)}_{3/1}$ $(r = 0, 1)$. The boundary value problem (1.13) was solved for obtaining $x_z$, $x^{(r)}_{1/1}$, and $x^{(r)}_{3/1}$. Its solutions in region $E$ which for $e \ll 1$ coincide with solutions $x_z$, $x^{(r)}_{1/1}$, and $x^{(r)}_{3/1}$, and are shown in Fig. 1, where the dependence of the initial velocity $\dot{x}(0)$ on $e$ can be seen for various values of $\mu$. Note that for $-10 \leqslant \mu < 1$ and $e \ll 1$ there is a single solution $x_z$; for $1 < \mu < 9$ and $e \ll 1$ there are three solutions $x_z$, $x^{(0)}_{1/1}$, and $x^{(1)}_{1/1}$ that satisfy the inequalities $\dot{x}^{(1)}_{1/1}(0, e) < \dot{x}_z(0, e) < \dot{x}^{(0)}_{1/1}(0, e)$, and for $9 < \mu \leqslant 10$ and $e \ll 1$ there are five solutions $x_z$, $x^{(r)}_{1/1}$, and $x^{(r)}_{3/1}$ $(r = 0, 1)$ for which $\dot{x}^{(1)}_{1/1}(0, e) < \dot{x}^{(1)}_{3/1}(0, e) < \dot{x}_z(0, e) < \dot{x}^{(0)}_{3/1}(0, e) < \dot{x}^{(0)}_{1/1}(0, e)$.

The dependence of the initial velocity of calculated solutions on parameters $e$ and $\mu$ may be specified in the form of surface $S$ in the space $(\alpha = \dot{x}(0), e, \mu)$. It should be noted that several values of $\dot{x}(0)$ may correspond to the same values of $e$ and $\mu$. The curves in Fig. 1 represent the intersection of surface $S$ with planes $\mu = \text{const}$. The orthogonal projection of that surface on the plane $(e, \mu)$ defines the subdivision of region $E$ in subregions such that at all points of a single subregion there is the same number of solutions of the boundary value problem (1.3). The curves that produce this subdivision are called branching curves, and are shown in Fig. 2 (the meaning of curves $\gamma_2^-$ and of other similar notation will be explained later). The points of surface $S$ at which the plane tangent to it is parallel to the $\alpha$-axis are projected on the branching curves. The projection image in any neighborhood of

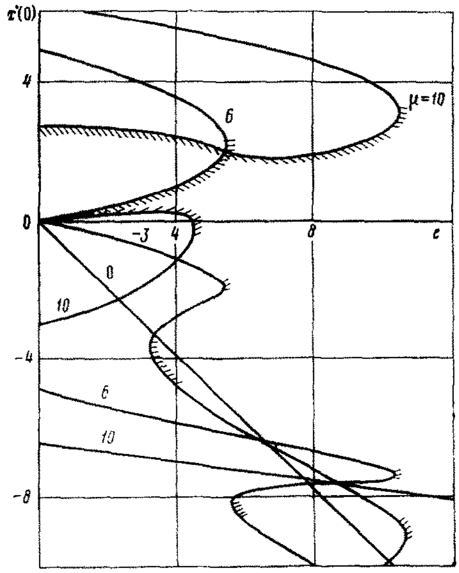


Fig. 1

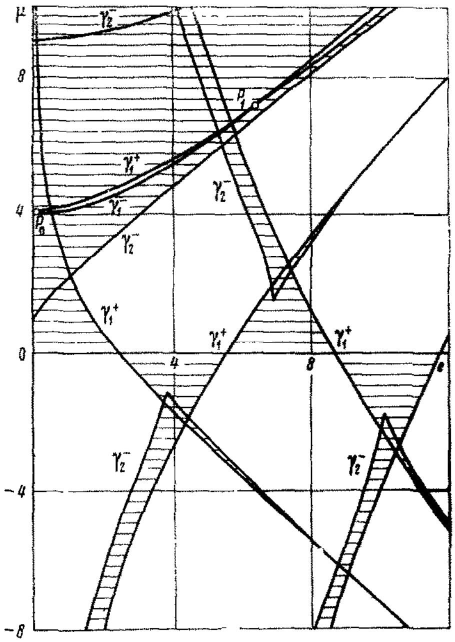


Fig. 2

such points is not one-to-one. All remaining points of $S$ have a neighborhood in which image is one-to-one. Solutions of the boundary value problem (1.3) are also completely determined by the quantity $x(\pi/2)$. In the space $(\bar{\alpha} = x(\pi/2), e, \mu)$ the dependence of $x(\pi/2)$ on $e$ and $\mu$ for the derived solutions may be specified in the form of surface $\bar{S}$ diffeomorphic of surface $S$. Below, if this does not present difficulties, the solutions which are continuations of solutions $x_{n/1}^{(r)}$, $\bar{x}_{n/1}^{(r)}$, and $x_z$ in the region of large $e$, are denoted by $x_{n/1}^{(r)}$, $\bar{x}_{n/1}^{(r)}$, and $x_z$, respectively.

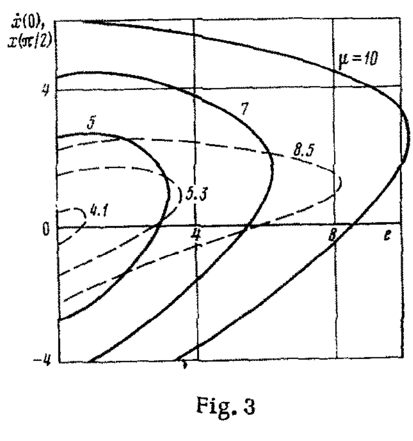


Fig. 3

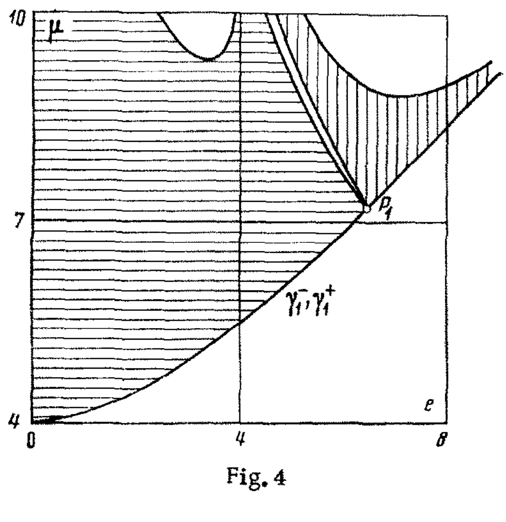


Fig. 4

Solutions $x_{2/1}^{(0)}$ and $x_{2/1}^{(1)}$ were obtained by solving the boundary value problem (2.2). The dependence of the initial velocity $x^{\cdot}(0)$ of these solutions on $e$ for various values of $\mu$ are shown in Fig. 3 by solid lines. The solution for which $x^{\cdot}(0) > 0$ ($x^{\cdot}(0) < 0$) with $e \ll 1$ is $x_{2/1}^{(0)}$ and $x_{2/1}^{(1)}$. In the space $(\alpha, e, \mu)$ surface $S'$ corresponds to solutions $x_{2/1}^{(0)}$ and $x_{2/1}^{(1)}$.

Some properties of surfaces $S$ and $S'$ are indicated below. Let $x = X(t, \alpha, e, \mu)$ be the solution of Eq. (1.1) with initial conditions $X(0, \alpha, e, \mu) = 0$ and $X^{\cdot}(0, \alpha, e, \mu) = \alpha$. Then, if $(\alpha, e, \mu) \in S$ we have $X^{\cdot}(\pi, \alpha, e, \mu) = -\alpha$; if $(\alpha, e, \mu) \in S'$, then also $(-X^{\cdot}(\pi, \alpha, e, \mu), e, \mu) \in S'$. If $(\alpha, e, \mu) \in S'$, but $(\alpha, e, \mu) \overline{\in} S$, then points $(\alpha, e, \mu)$ and $(-X^{\cdot}(\pi, \alpha, e, \mu), e, \mu)$ lie on surface $S'$ on different sides of $S$. One of these points corresponds to solution $x_{2/1}^{(0)}$, and the other to solution $x_{2/1}^{(1)}$. For some $e = e_*(\mu)$ solutions $x_{2/1}^{(0)}$ and $x_{2/1}^{(1)}$ merge: $x_{2/1}^{(0)}(t, e_*) = x_{2/1}^{(1)}(t, e_*)$. Hence by virtue of (2.1) $x_{2/1}^{(0)}(t + \pi, e_*) = -x_{2/1}^{(0)}(t, e_*)$, and solution $x_{2/1}^{(0)}(t, e_*)$ satisfies the boundary conditions (1.3). Analysis of calculations shows (cf. Figs. 1 and 3) that surface $S'$ intersects surface $S$ along the merging line of solutions $x_{2/1}^{(0)}$ and $(x_{2/1}^{(1)})$. At points of that line the plane tangent to surface $S'$ is parallel to the $\alpha$-axis. The projection of line $S \cap S'$ on surface $(e, \mu)$ is the branching of solutions $x_{2/1}^{(0)}$ and $x_{2/1}^{(1)}$. It is shown in Fig. 2 by the curve $\gamma_1^-$.

Solutions $\bar{x}_{2/1}^{(0)}$ and $\bar{x}_{2/1}^{(1)}$ were obtained by solving the boundary value problem (2.3). Dependence of the initial coordinate of these solutions of $x(\pi / 2)$ on $e$ is shown in Fig. 3 by dash lines for several values of $\mu$. The solution for which $x(\pi / 2) > 0$ ($x(\pi / 2) < 0$) with $e \ll 1$ is $\bar{x}_{2/1}^{(1)}$ ($\bar{x}_{2/1}^{(0)}$). In the space $(\bar{\alpha}, e, \mu)$ surface $\bar{S}'$ corresponds to solutions $\bar{x}_{2/1}^{(0)}$ and $\bar{x}_{2/1}^{(1)}$. The properties of $\bar{S}$ and $\bar{S}'$ are analogous to those of surface $S$ and $S'$. The projection of line $\bar{S} \cap \bar{S}'$ on plane $(e, \mu)$ is the curve of branching of solutions $\bar{x}_{2/1}^{(0)}$, $\bar{x}_{2/1}^{(1)}$ and is shown in Fig. 2 by curve $\gamma_1^+$. Although in the scale selected for Fig. 2 curves $\gamma_1^+$ and $\gamma_1^-$ should merge, they are shown separately for clarity, with curve $\gamma_1^-$ shown in the correct position.

3. **Stability of the $2\pi$-periodic solutions.** The variational equation for Eq. (1.1) is of the form

$$y^{\cdot\cdot} + \mu y \cos x = 0 \tag{3.1}$$

Let in (3.1) $x = x(t)$ be the periodic solution of Eq. (1.1) such that function $\cos x(t)$ is of period $T$. The characteristic equation of (3.1) is then of the form

$$\rho^2 - 2A\rho + 1 = 0, \quad A = {}^1/_2 \, [y_1(T) + y_2^{\cdot}(T)]$$

where $y_1(t)$ and $y_2(t)$ are solutions of Eq. (3.1) with initial conditions $y_1(0) = y_2^{\cdot}(0) = 1$, $y_1^{\cdot}(0) = y_2(0) = 0$. If $|A| < 1$, the necessary condition of stability of solution $x(t)$ is satisfied. In that case we say that $x(t)$ is stable in linear approximation. When $|A| > 1$, solution $x(t)$ is unstable. Throughout the subsequent analysis stability is understood to mean stability in linear approximation.

The stability region boundary is specified by the equations $A = 1$ and $A = -1$. For $A = 1$ Eq. (3.1) has a nontrivial periodic solution of period $T$. If

$$\operatorname{rank}\begin{Vmatrix} y_1(T)-1 & y_2(T) \\ y_1^{\cdot}(T) & y_2^{\cdot}(T)-1 \end{Vmatrix} = 1 \tag{3.2}$$

this solution is unique and accurate to the constant co-factor. If condition (3.2) is not satisfied, all solutions of Eq. (3.1) are $T$-periodic. When $A = -1$ Eq. (3.1) has a nontrivial $T$-antiperiodic solution. If

$$\operatorname{rank}\begin{Vmatrix} y_1(T)+1 & y_2(T) \\ y_1^{\cdot}(T) & y_2^{\cdot}(T)+1 \end{Vmatrix} = 1 \tag{3.3}$$

that solution is unique. Otherwise all solutions of Eq. (3.1) are $T$-antiperiodic. Let $\cos x(t)$ be an even function and $|A| = 1$. Then when conditions (3.2) or (3.3) are satisfied, the respective solution of Eq. (3.1) is either even or odd.

If $x(t)$ is a solution of the boundary value problem (1.3), then $\cos x(t)$ is an even $\pi$-periodic function. Hence it is possible to assume $T = \pi$ when investigating stability of solutions $x_z$, $x_{1/_1}^{(r)}$, $x_{1/_1}^{(r)}$ $(r = 0, 1)$ The stability region of these solutions is represented by sections of surface $S$ bounded by curves along which $|A| = 1$. We denote these curves by $\Gamma_m^+$, $\Gamma_m^-$ $(m = 1, 2)$. Along the curve $\Gamma_m^+$ $(\Gamma_m^-)$ Eq. (3.1) has an even (odd) $\pi m$-periodic solution. Thus along curves $\Gamma_1^+$ and $\Gamma_1^-$ we have $A = 1$, and along curves $\Gamma_2^+$ and $\Gamma_2^-$ $A = -1$. The construction of curves $\Gamma_m^+$ and $\Gamma_m^-$ reduces to the solution of the following boundary value problems for the system of Eqs. (1.1) and (3.1):

$$\begin{aligned} &x(0) = x^{\cdot}\left(\frac{\pi}{2}\right) = y^{\cdot}(0) = y^{\cdot}\left(\frac{\pi}{2}\right) = 0 \quad (\Gamma_1^+) \\ &x(0) = x^{\cdot}\left(\frac{\pi}{2}\right) = y(0) = y\left(\frac{\pi}{2}\right) = 0 \quad (\Gamma_1^-) \\ &x(0) = x^{\cdot}\left(\frac{\pi}{2}\right) = y^{\cdot}(0) = y\left(\frac{\pi}{2}\right) = 0 \quad (\Gamma_2^+) \\ &x(0) = x^{\cdot}\left(\frac{\pi}{2}\right) = y(0) = y^{\cdot}\left(\frac{\pi}{2}\right) = 0 \quad (\Gamma_2^-) \end{aligned} \tag{3.4}$$

where the symbol in parentheses defines the curve obtained by solving the respective boundary value problem. We denote, respectively, by $\gamma_m^+$ and $\gamma_m^-$ the projections of curves $\Gamma_m^+$ and $\Gamma_m^-$ on the plane $(e, \mu)$. The disposition of curves $\gamma_m^+$ and $\gamma_m^-$ in region $E$ appears in Fig. 2. Curves $\gamma_2^+$ are absent. The points of intersection of curves $\gamma_1^+$ with the axis $\mu = 0$ are determined by the equation $J_0(e) = 0$ where $J_0(e)$ is a Bessel function of the first kind of zero order. The subsequent investigation of curves $\gamma_1^+$ and $\gamma_1^-$ emanating from point $P_0$ $(e = 0, \mu = 4)$ which intersect at point $P_1$ $(e = 6.43, \mu = 7.17)$ and for $e \ll 1$ are specified by the equations

$$\mu = 4 + \frac{1}{9}e^2 - \frac{41}{38880}e^4 + O(e^6) \quad (\gamma_1^+)$$

$$\mu = 4 + \frac{1}{9}e^2 - \frac{59}{38880}e^4 + O(e^6) \quad (\gamma_1^-)$$

Curves $\gamma_1^+$, $\gamma_1^-$, and $\gamma_2^-$ represent the boundaries of projections of stability regions of solutions $x_z$, $x^{(r)}_{s/1}$ and $x^{(r)}_{s/1}$ on the plane $(e, \mu)$. Projections of these regions are shown shaded in Fig. 2. Several of the considered solutions exist for some $e$ and $\mu$. Sections of stability of these solutions are shaded in Fig. 1.

The stability of solutions $x^{(r)}_{s/1}$ and $\bar{x}^{(r)}_{s/1}$ is analyzed similarly. By virtue of the last two of formulas (2.1)

$$\cos x^{(0)}_{s/1}(t+\pi, e) = \cos x^{(1)}_{s/1}(t, e), \quad \cos \bar{x}^{(0)}_{s/1}(t+\pi, e) = \cos \bar{x}^{(1)}_{s/1}(t, e)$$

hence solutions $x^{(0)}_{s/1}$, $x^{(1)}_{s/1}$ ($\bar{x}^{(0)}_{s/1}$, and $\bar{x}^{(1)}_{s/1}$) are at the same time stable or unstable. The region of stability of solutions $x^{(0)}_{s/1}$, $x^{(1)}_{s/1}$, ($\bar{x}^{(0)}_{s/1}$, and $\bar{x}^{(1)}_{s/1}$) appear in Fig. 4 hatched by horizontal (vertical) lines.

4. **Branching of $2\pi$-periodic solutions.** Using function $X(t, \alpha, e, \mu)$ from Sect. 2 above we can write the equation of surface $S$ in the space $(\alpha, e, \mu)$ in the form $X^{\cdot}(\pi/2, \alpha, e, \mu) = 0$. The curves on surface $S$ at points at which the plane tangent to $S$ is parallel to the $\alpha$-axis are specified by the equations

$$X^{\cdot}(\pi/2, \alpha, e, \mu) = 0, \quad \partial X^{\cdot}(\pi/2, \alpha, e, \mu)/\partial\alpha = 0 \tag{4.1}$$

The projections of these curves on the plane $(e, \mu)$ are branching curves of the boundary value problem (1.3). Along curves (4.1) functions $x = X(t, \alpha, e, \mu)$ and $y = \partial X(t, \alpha, e, \mu)/\partial\alpha$ satisfy the last of conditions (3.4), hence these branching curves are the $\gamma_2^-$-curves. It can be shown in the same way that projection of the line of intersection $S$ and $S'$ ($\bar{S}$, $\bar{S}'$) on the plane $(e, \mu)$ is the curve $\gamma_1^-$ ($\gamma_1^+$).

Branching of the $2\pi$-periodic solutions of Eq. (1.1) on curves $\gamma_2^-$ is similar to the branching of the $2\pi$-periodic solutions of the equation considered in [6], and are not considered here. We shall only analyze the branching of the $2\pi$-periodic solutions on curves $\gamma_1^+$ and $\gamma_1^-$ emanating from point $P_0$. Let point $(\alpha_*, e_*, \mu_*)$ lie on curve $\Gamma_1^+$ or $\Gamma_1^-$ and $x_*(t) = X(t, \alpha_*, e_*, \mu_*)$. Then using the notation

$$q = x - x_*(t), \quad \varepsilon = e - e_*, \quad \delta = \mu - \mu_* \quad f(t) = \mu_* \cos x_*(t)$$

$$H(q, t, \varepsilon, \delta) = \varepsilon \sin t + \mu_* \sin x_*(t) - (\mu_* + \delta) \sin [x_*(t) + q] + \mu_* q \cos x_*(t)$$

Eq. (1.1) can be written as

$$q^{\cdot\cdot} + f(t) q = H(q, t, \varepsilon, \delta) \tag{4.2}$$

where

$$f(t+\pi) = f(t), \quad H(q, t+\pi, \varepsilon, \delta) = -H(-q, t, \varepsilon, \delta) \tag{4.3}$$

$$f(t) = f(-t), \quad H(q, t, \varepsilon, \delta) = -H(-q, -t, \varepsilon, \delta) \tag{4.4}$$

Function $H(q, t, \varepsilon, \delta)$ is analytic with respect to $q$, $\varepsilon$, and $\delta$ at point $q = \varepsilon = \delta = 0$, and $H(q, t, \varepsilon, \delta) = O(q^2 + |\varepsilon| + |\delta|)$. Investigation of the $2\pi$-periodic solutions of Eq. (1.1) which for $e = e_*$ and $\mu = \mu_*$ are transformed into solution $x_*(t)$ is equivalent to the investigation of the $2\pi$-periodic solutions of Eq. (4.2) which vanish when $\varepsilon = \delta = 0$. Such investigation is carried out differently depending on the fulfilment of condition (3. 2). Calculations show that that condition is violated only at points $P_0$ and $P_1$.

Let us consider an arbitrary point of curves $\gamma_1^+$ and $\gamma_1^-$ different from points $P_0$ and $P_1$. At such point the linearly independent solutions of the equation

$$u^{\cdot\cdot} + f(t)u = 0 \tag{4.5}$$

may be written as

$$u_1(t) = u(t), \quad u_2(t) = u(t)\,t + v(t) \tag{4.6}$$

where $u(t)$ and $v(t)$ are $\pi$-periodic functions with $u(t)$ even on curve $\gamma_1^+$ and odd on curve $\gamma_1^-$. Let us consider the auxilliary system

$$q^{\cdot\cdot} + f(t)q = H(q, t, \varepsilon, \delta) - pu(t), \quad \int_0^{2\pi} u(t)\,q\,dt = a$$

where $q$ is the unknown function, $p$ the unknown constant, and $a$ is an arbitrary constant. For reasonably small $|a|$, $|\varepsilon|$, and $|\delta|$ this system has the unique $2\pi$-periodic solution with respect to $t$ [3]

$$q = q_*(t, a, \varepsilon, \delta), \quad p = p_*(a, \varepsilon, \delta) \tag{4.7}$$

analytically dependent on $a$, $\varepsilon$, and $\delta$ and satisfying conditions $q_*(t, 0, 0, 0) = 0$, and $p_*(0, 0, 0) = 0$. Determination of the $2\pi$-periodic solutions of Eq. (4. 2) that vanish for $\varepsilon = \delta = 0$ is equivalent to finding the roots $a = a(\varepsilon, \delta)$ of the equation [3]

$$p_*(a, \varepsilon, \delta) = 0 \tag{4.8}$$

such that $a(0, 0) = 0$. Let $a(\varepsilon, \delta)$ be the root of Eq. (4. 8) and $a(0, 0) = 0$. Then $q = q_*[t, a(\varepsilon, \delta), \varepsilon, \delta]$ is a $2\pi$-periodic solution of Eq. (4. 2). The characteristic indices $\lambda$ of that equation are of the form

$$\lambda^2 = \frac{M^2}{2\pi W}\frac{\partial p_*[a(\varepsilon, \delta), \varepsilon, \delta]}{\partial a}[1 + o(1)]$$

$$M = \int_0^{2\pi} u^2(t)\,dt, \quad W = u^2 + u^{\cdot}v - uv^{\cdot} = \text{const}$$

where $o(1)$ denotes function of $\varepsilon$, $\delta$ that tends to vanish as $\varepsilon \to 0$ and $\delta \to 0$, and $W$ is the Wronskian of functions (4. 6).

We shall indicate some of the properties of solution (4. 7). Using formulas (4. 3) and the $\pi$-periodicity of function $u(t)$ we can show that

$$q_*(t + \pi, a, \varepsilon, \delta) = -q_*(t, -a, \varepsilon, \delta), \quad p_*(a, \varepsilon, \delta) = -p_*(-a, \varepsilon, \delta) \tag{4.9}$$

By virtue of (4. 4) and the properties of evenness of function $u(t)$ we have

$$q_*(t, a, \varepsilon, \delta) = -q_*(-t, a, \varepsilon, \delta), \quad (\alpha_*, e_*, \mu_*) \in \Gamma_1^- \tag{4.10}$$

$$q_*(t, a, \varepsilon, \delta) = -q_*(-t, -a, \varepsilon, \delta), \quad (\alpha_*, e_*, \mu_*) \in \Gamma_1^+ \tag{4.11}$$

It follows from (4. 9) that $p_*(a, \varepsilon, \delta) = a\varphi(a^2, \varepsilon, \delta)$, where $\varphi(z, \varepsilon, \delta)$ is an analytic function of $z, \varepsilon, \delta$ at point $z = \varepsilon = \delta = 0$, $\varphi(0, 0, 0) = 0$. Equation (4. 8) has the trivial solution $a = 0$ to which corresponds the odd $\pi$-antiperiodic (cf. (4. 9)—

(4.11) solution of Eq. (4.2) $q_0(t, \varepsilon, \delta) = q_*(t, 0, \varepsilon, \delta)$ that satisfies the condition

$$(x_*^{\cdot}(0) + q_0^{\cdot}(0, \varepsilon, \delta), e_* + \varepsilon, \mu_* + \delta) \in S \tag{4.12}$$

For $a \neq 0$ Eq. (4.8) becomes

$$\varphi\,(a^2, \varepsilon, \delta) = 0 \tag{4.13}$$

Let us represent function $\varphi\,(z, \varepsilon, \delta)$ in the form $\varphi\,(z, \varepsilon, \delta) = \varphi_{100}z + \varphi_{010}\varepsilon + \varphi_{001}\delta + O\,(z^2 + \varepsilon^2 + \delta^2)$. Calculations show that at all points of curves $\gamma_1^+$ and $\gamma_1^-$ (including points $P_0$ and $P_1$ ) $\varphi_{010}^2 + \varphi_{001}^2 > 0$, $\varphi_{100} \neq 0$. For fairly small $|a|$, $|\varepsilon|$, and $|\delta|$, by the theorem on implicit function Eq. (4.13) is equivalent to the equation

$$a^2 = h\,(\varepsilon, \delta) \tag{4.14}$$

where $h\,(\varepsilon, \delta) = -\varphi_{100}^{-1}\,(\varphi_{010}\varepsilon + \varphi_{001}\delta) + O\,(\varepsilon^2 + \delta^2)$ is an analytic function of $\varepsilon$, $\delta$ at point $\varepsilon = \delta = 0$. Since $\varphi_{010}^2 + \varphi_{001}^2 > 0$ , hence point $\varepsilon = \delta = 0$ is not a singular point of curve $h\,(\varepsilon, \delta) = 0$. In region $\{\varepsilon, \delta : h\,(\varepsilon, \delta) > 0\}$ Eq. (4.14) has two real roots $a_1(\varepsilon, \delta) = [h\,(\varepsilon, \delta)]^{1/2}$ and $a_2(\varepsilon, \delta) = -a_1\,(\varepsilon, \delta)$ to which correspond solutions $q_j\,(t, \varepsilon, \delta) = q_*\,[t, a_j\,(\varepsilon, \delta), \varepsilon, \delta]$ $(j = 1, 2)$ which are linked by the relationship $q_2\,(t, \varepsilon, \delta) = -\,q_1\,(t + \pi, \varepsilon, \delta)$.

If $(\alpha_*, e_*, \mu_*) \in \Gamma_1^-$, then $q_j\,(t, \varepsilon, \delta) = -\,q_j\,(-\,t, \varepsilon, \delta)$ and

$$(x_*^{\cdot}\,(0) + q_j^{\cdot}\,(0, \varepsilon, \delta), e_* + \varepsilon, \mu_* + \delta) \in S' \quad (j = 1, 2)$$

If $(\alpha_*, e_*, \mu_*) \in \Gamma_1^+$, then $q_j\,(-\,t + \pi\,/\,2, \varepsilon, \delta) = q_j\,(t + \pi\,/\,2, \varepsilon, \delta)$ and

$$\left(x_*\left(\frac{\pi}{2}\right) + q_j\left(\frac{\pi}{2}, \varepsilon, \delta\right), e_* + \varepsilon, \mu_* + \delta\right) \in S' \quad (j = 1, 2)$$

The characteristic indices of solutions $q_0$, $q_1$, and $q_2$ are of the form

$$\lambda^2 = -\frac{M^2\varphi_{100}}{2\pi W}\,h\,(\varepsilon, \delta)[1 + o\,(1)] \quad (q_0)$$

$$\lambda^2 = \frac{M^2\varphi_{100}}{\pi W}\,h\,(\varepsilon, \delta)[1 + o\,(1)] \quad (q_1, q_2)$$

This implies that the curve $\gamma_1^+$ or $\gamma_1^-$ in the plane $(\varepsilon, \delta)$ is defined by the equation $h\,(\varepsilon, \delta) = 0$, and is the boundary of the stability region of solution $q_0$ and the branching curve of solutions $q_1$ and $q_2$ . Along that curve $q_0 = q_1 = q_2$. Solutions $q_1$ and $q_2$ can be stable or unstable. When they are stable (unstable), then in the region of existence of these solutions solution $q_0$ is unstable (stable). The Wronskian of functions (6.6) passes at point $P_1$ through infinity and changes its sign. The obtained results make it possible to check the numerical investigation of stability of solutions $x_z$, $x_{s/1}^{(r)}$, $\bar{x}_{s/1}^{(r)}$ $(r = 0, 1)$ in the vicinity of curves $\gamma_1^+$ and $\gamma_1^-$ (cf. the position of shaded regions in Figs. 2, 4, and 5).

We shall now investigate the branching of the $2\pi$-periodic solutions in the neighborhood of points $P_0$ and $P_1$ at which the linearly independent solutions $u_1\,(t)$ and $u_2\,(t)$ of Eq. (4.5) are $\pi$-periodic. Let us assume that $u_1\,(t)$ is even and $u_2\,(t)$ odd, and consider the auxilliary system

$$q^{\cdot\cdot} + f\,(t)\,q = H\,(q, t, \varepsilon, \delta) - p_1u_1\,(t) - p_2u_2\,(t)$$

$$\int_0^{2\pi} u_1\,(t)\,q\,dt = a_1, \quad \int_0^{2\pi} u_2\,(t)\,q\,dt = a_2$$

where $q$ is the unknown function, $p_1$ and $p_2$ the unknown constants, and $a_1$ and $a_2$ are arbitrary constants. For reasonably small $|a_1|$, $|a_2|$, $|\varepsilon|$, and $|\delta|$ this system has the unique $2\pi$-periodic with respect to $t$ solution [3]

$$q = \bar{q}_*(t, a_1, a_2, \varepsilon, \delta) \tag{4.15}$$

$$p_1 = p_1^*(a_1, a_2, \varepsilon, \delta), \quad p_2 = p_2^*(a_1, a_2, \varepsilon, \delta)$$

which analytically depends of $a_1, a_2, \varepsilon, \delta$ and satisfies conditions $\bar{q}_*(t, 0, 0, 0, 0) = 0$, and $p_j^*(0, 0, 0, 0) = 0$ $(j = 1, 2)$. The determinations of the $2\pi$-periodic solutions of Eq. (4.2) which vanish for $\varepsilon = \delta = 0$ is similar to the determination of roots $a_1 = a_1(\varepsilon, \delta)$ and $a_2 = a_2(\varepsilon, \delta)$ of system [3]

$$p_1^*(a_1, a_2, \varepsilon, \delta) = 0, \quad p_2^*(a_1, a_2, \varepsilon, \delta) = 0 \tag{4.16}$$

such that $a_1(0, 0) = a_2(0, 0) = 0$.

The following relationships:

$$\begin{aligned} &\bar{q}_*(t + \pi, a_1, a_2, \varepsilon, \delta) = -\bar{q}_*(t, -a_1, -a_2, \varepsilon, \delta) \\ &\bar{q}_*(t, a_1, a_2, \varepsilon, \delta) = -\bar{q}_*(-t, -a_1, a_2, \varepsilon, \delta) \\ &p_j^*(a_1, a_2, \varepsilon, \delta) = -p_j^*(-a_1, -a_2, \varepsilon, \delta) \quad (j = 1, 2) \\ &p_1^*(a_1, a_2, \varepsilon, \delta) = -p_1^*(-a_1, a_2, \varepsilon, \delta) \\ &p_2^*(a_1, a_2, \varepsilon, \delta) = p_2^*(-a_1, a_2, \varepsilon, \delta) \end{aligned} \tag{4.17}$$

whose proof is similar to that of formulas (4.9) — (4.11), are valid for solutions (4.15). By virtue of (4.17)

$$p_j^*(a_1, a_2, \varepsilon, \delta) = a_j\varphi_j(a_1^2, a_2^2, \varepsilon, \delta) \quad (j = 1, 2)$$

where $\varphi_j(z_1, z_2, \varepsilon, \delta)$ are analytic functions of $z_1, z_2, \varepsilon, \delta$ at point $z_1 = z_2 = \varepsilon = \delta = 0$, and $\varphi_j(0, 0, 0, 0) = 0$. System (4.16) has the trivial roots $a_1 = a_2 = 0$ to which corresponds the odd $\pi$-antiperiodic solution of Eq. (4.2) $q_0(t, \varepsilon, \delta) = \bar{q}_*(t, 0, 0, \varepsilon, \delta)$ which satisfies condition (4.12).

At point $P_0$ we have $x_*(t) = 0$ and $H(q, t + \pi, \varepsilon, \delta) = H(q, t, -\varepsilon, \delta)$. By virtue of the last equality $\bar{q}_*(t + \pi, a_1, a_2, \varepsilon, \delta) = \bar{q}_*(t, a_1, a_2, -\varepsilon, \delta)$ and, consequently, $q_0(t, \varepsilon, \delta) = -q_0(t, -\varepsilon, \delta)$, $q_0(t, 0, \delta) = 0$. Thus $q_0 = x_z$ for $\delta \neq 0$. Solution $q_0$ is the continuation of solution $x_z$ to the removable singular point $P_0$.

Without presenting the general analysis of system (4.16) we shall indicate its roots which correspond to solutions $x_{2/1}^{(r)}$ and $\bar{x}_{2/1}^{(r)}$ $(r = 0,1)$. For $a_1 = a \neq 0$ and $a_2 = 0$ that system is transformed into the equation

$$\varphi_1(a^2, 0, \varepsilon, \delta) = 0 \tag{4.18}$$

and function $q_*(t, a, \varepsilon, \delta) = \bar{q}_*(t, a, 0, \varepsilon, \delta)$ satisfies formula (4.11) by virtue of (4.17). Similarly system (4.16) is transformed for $a_1 = 0$ and $a_2 = a \neq 0$ into the equation

$$\varphi_2(0, a^2, \varepsilon, \delta) = 0 \tag{4.19}$$

and equality (4.10) holds for function $q_*(t, a, \varepsilon, \delta) = \bar{q}_*(t, 0, a, \varepsilon, \delta)$. In these cases the branching of periodic solutions is similar to the branching of periodic solutions at points of curves $\gamma_1^+$ and $\gamma_1^-$ not coincident with points $P_0$ and $P_1$. Solutions of Eq. (1.1) that correspond to roots of Eqs. (4.18) and (4.19) are, respectively, the solutions $\bar{x}_{2/1}^{(r)}$, $x_{2/1}^{(r)}$.

Above, we have investigated curves $\gamma_1^+$ and $\gamma_1^-$ emanating from point $P_0$. It can be similarly shown that the remaining curves $\gamma_1^+$ (Fig. 2) are branching curves of the $2\pi$-periodic solutions of Eq. (1.1) whose numerical determination reduces to solving the boundary value problem (2.3) which has no solution for $e \ll 1$.

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